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SIMULTANEOUS LIFE TESTING IN A RANDOM ENVIRONMENT

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Examined here is a class of multivariate lifetime distributions generated by a physical model in which a group of like devices is simultaneously exposed to a random wear or damage environment. This random wear is represented by a nonnegative stochastic process with independent increments. Associated with each device is a random threshold and the device fails when the wear attains this threshold. It is shown that tied failure times occur with positive probability. Algorithms are developed to obtain the probabilistic properties of various random variables associated with the joint failure time vector. In particular, these algorithms are used to find the probability of obtaining a specific tie configuration and the large sample behavior of the number of distinct failure times.

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1. Introduction

Several authors have proposed models for the lifetime of a device exposed to a wear or damage environment. Esary, Marshall and Proschan [5] have studied the life distributions generated by a shock model in which the environmental wear consists of shocks arriving according to a homogeneous Poisson process and the device fails when the cumulative damage caused by these shocks exceeds some threshold level. A-Hameed and Proschan [1] extend these results to include nonhomogeneous Poisson processes. Gaver [7] has proposed a model in which the hazard function is a nonnegative stochastic process with independent increments. Typical examples for which a wear model might be appropriate include a sensitive electrical component which may occasionally experience damaging current surges and aircraft components which receive environmental stress due to atmospheric turbulence as well as take-off and landing shocks. Other examples appear in the references.

In life testing situations, one commonly tests identical components independently and then uses the i.i.d. failure times to make inferences concerning the life distribution. However, even with a simple shock model in which shocks are arriving according to a homogeneous Poisson process with intensity λ and each shock causes an identical amount of damage δ, the parameters λ and δ are not identifiable with i.i.d. failure times. Also, the component to be tested may be only a small part of a much larger system (e.g. a guidance component in a missile) and so the cost of testing each component separately may be very large compared with the cost of that component. In addition, the testing procedure itself may be very time-consuming. In these cases considerable savings in time and money may be achieved by testing all components simultaneously, i.e. all components simultaneously experience the same wear environment. Reynolds and Savage [12] have obtained estimates with simultaneous testing of a time-scale parameter for a particular shock model in which each shock causes a known constant amount of damage. The aim of this paper is to develop the probabilistic properties of the joint failure time vector obtained by simultaneous testing which will enable statistical inference procedures to be developed for a broad class of wear models and wear processes. In addition these results can be used to model the behavior of systems which are composed of several components operating independently, all of which experience the same damage environment (e.g., a guidance system might contain three separate gyroscopes operating independently; an unmanned satellite may be controlled by a computer with two or three identical back-up computers). The main results are contained in Algorithms 1, 2 and 3 which give simple computational formulas for the expectation of various random variables associated with the joint failure time vector.

2. Wear models

Esary, Marshall and Proschan [5] describe a model in which a device is exposed to environmental wear or damage. The device fails when the wear attains a critical threshold ξ. The environmental wear is represented by a nonnegative stochastic process $X(t)$, $t \geq 0$, and ξ is in general assumed to be a r.v. with some d.f. G. If T represents the lifetime of a device, then the conditional survival probability given a realization of the wear process X is

$$\mathbf{P}(T > t \mid X(\cdot)) = \bar{G}(X(t)), \quad t \geq 0 \tag{1}$$

where $\bar{G} = 1 - G$, and so

$$\mathbf{P}(T > t) = \mathbf{E}\bar{G}(X(t)). \tag{2}$$

The models given by (1) and (2) are referred to as wear models.

Gaver [7] has discussed a wear model in which $\bar{G}(x) = \mathrm{e}^{-x}$, $x \geq 0$. He has shown that this model, referred to here as a Gaver model, provides a generalization of the loss of memory property of the exponential distribution. Conditioned on realizations of X, the probability that a device survives past time $t+s$ given that the device has survived past time t is stochastically independent of $X(t)$.

Two classes of wear processes are given by Definitions 1 and 2.

Definition 1. S is the class of all nondegenerate, separable stochastic processes $X(t)$, $t \geq 0$, such that

(i) X has independent increments and $X(0)=0$ a.s.,
(ii) X is centered with no fixed points of discontinuity,
(iii) $\mathbf{P}(X(t)<0)=0$, $t \geq 0$,
(iv) the sample paths of X are right-continuous a.s.

See [3, pp. 357, 408] for precise definitions of centering and fixed points of discontinuity. Processes in S have infinitely divisible increments and the Lévy representation of the characteristic functions of processes in S is given in [12, p. 231].

Definition 2. S_1 is the class of all processes X in S with moment generating function (m.g.f.) of the form

$$\log \mathbf{E}\, e^{-u[X(t)-X(s)]} = -[\alpha(t)-\alpha(s)] \int_{0^+}^{\infty} (1-e^{-uv})\, dQ(v), \qquad u \geq 0,\ 0 \leq s \leq t, \tag{3}$$

where

(i) $\alpha(0)=0$ and $\alpha(t) \to \infty$ as $t \to \infty$,
(ii) $\alpha(t)$ is continuous and nondecreasing,
(iii) $Q(v)$ is nondecreasing, $-\infty \leq Q(0) < 0$ and $Q(\infty)=0$,
(iv) $\int_{0^+}^{\infty} v/(1-v^2)\, dQ(v) < \infty$.

In S_1, the measure determined by $Q(v)$ is called the Lévy measure and $\alpha(t)$ is called the time-scale function. Throughout, it is assumed that the region of integration with respect to the Lévy measure is a subset of $(0, \infty)$.

Processes in S_1 include the Poisson process, for which $Q(v)=-1$, $0 \leq v \leq 1$, and $Q(v)=0$, $v \geq 1$, and the compound Poisson process, for which $F(v)=1+Q(v)$ is a d.f. on $(0,\infty)$. Also, there exists a process X in S_1 which has the gamma distribution

$$\mathbf{P}(X(t) \leq x) = \int_0^x [\Gamma(\alpha(t))\beta^{\alpha(t)}]^{-1} v^{\alpha(t)-1} e^{-v/\beta}\, dv$$

where $\beta > 0$ and α satisfies conditions (i) and (ii) of Definition 2. This process is called the gamma process and has Lévy measure $Q(v) = -\int_v^{\infty} x^{-1} e^{-x/\beta}\, dx$. Note that $Q(0)=-\infty$.

For X in S_1, define $M(u)=\int_0^{\infty}(1-e^{-uv})\, dQ(v)$, $u \geq 0$. Then $M(u)$ is a non-negative, continuous, nondecreasing function of u, and $\mathbf{E}\exp(-uX(t)) = \exp(-\alpha(t)M(u))$. Thus, in a Gaver model,

$$\mathbf{P}(T>t) = e^{-\alpha(t)M(1)}, \quad t \geq 0. \tag{4}$$

For any given Lévy measure it is easily seen from (4) that Gaver models can generate any continuous marginal lifetime distribution by choosing the appropriate time-scale function α. Esary, Marshall and Proschan [5, Theorem 5.1] show that the marginal distributions generated by wear models are exponential for all X in S_1 with stationary increments iff G is exponential.

If n identical devices are simultaneously exposed to the wear process and if the thresholds of the devices are i.i.d. given a realization of the process, then the joint lifetime distribution generated by the wear model is given by

$$\mathbf{P}(T_1>t_1,\dots,T_n>t_n \mid X(\cdot))=\prod_{j=1}^{n}\bar{G}(X(t_j)) \tag{5}$$

and

$$\mathbf{P}(T_1>t_1,\dots,T_n>t_n)=\mathbf{E}\prod_{j=1}^{n}\bar{G}(X(t_j)). \tag{6}$$

3. Properties of the joint failure time vector

Throughout this section it is assumed that the wear process belongs to S_1. A compound Poisson wear process corresponds to a physical situation in which shocks arrive according to a Poisson process and the damage caused by each shock is random. Since the sample paths of a gamma process are constant except for a countable number of jumps and since the jump times form a countable dense set in $(0,\infty)$, see [9, p. 8], then this process corresponds to a situation in which a device is almost continually experiencing shocks. However, the damage caused by each shock is relatively small. Since the sample paths of processes in S_1 are constant except for at most a countable number of jumps, then the device can only fail at a jump (shock).

When n devices are simultaneously exposed to a single realization of the wear process, it is possible to have more than one device fail at a jump. This is formalized in Theorem 1. It is assumed throughout this section that, given a realization of the wear process, the random thresholds of the n devices are mutually independent with common d.f. G. For a set of failure times $\boldsymbol{t}=(t_1,\dots,t_n)$, let k_n denote the number of distinct failure times. For convenience, k will often be used in place of k_n. When the number of distinct failure times is to be treated as a random variable, then K_n and K are used. Define $D(\boldsymbol{t})=(\tau_1,\dots,\tau_k)$ to be the *distinct ordering* of $\boldsymbol{t}$, where $\tau_0=0$ and $\tau_j=\min_{1\le i\le n}(t_i : t_i>\tau_{j-1})$, $1\le j\le k$. Also, define $C(\boldsymbol{t})=(c_1,\dots,c_k)$ to be the *tie configuration* of $\boldsymbol{t}$, where c_j is the number of failures at τ_j. Let $c_{k+j}=0$, $j\ge 1$, and $s_j=\sum_{i=j}^{\infty}c_i$. Then from (6), the joint lifetime distribution generated by a Gaver model is

$$\mathbf{P}(T_1>t_1,\dots,T_n>t_n)=\prod_{j=1}^{k}\exp(-[\alpha(\tau_j)-\alpha(\tau_{j-1})]M(s_j)). \tag{7}$$

From (7) it can be seen that K_n, $D(\boldsymbol{t})$, $C(\boldsymbol{t})$ are sufficient statistics for the distribution of the failure times $\boldsymbol{t}$.

Two preliminary lemmas are required before proceeding to the main results of this section. Define $t_{j,e}$ to be the time of the jth jump of X which is greater than or equal to

$\varepsilon > 0$, and define $v_{j,\varepsilon}$ to be the size of that jump. Define $X(t-) = \lim_{\delta \downarrow 0} X(t-\delta)$ and let $q_\varepsilon = -Q(\varepsilon) > 0$, $\varepsilon > 0$.

Lemma 1. *If $X \in S_1$, then for all $\varepsilon > 0$ such that $q_\varepsilon > 0$*

(i) *$\Delta\alpha_{j,\varepsilon} = \alpha(t_{j,\varepsilon}) - \alpha(t_{j-1,\varepsilon})$, $j \geq 1$, are i.i.d. exponential r.v.'s with mean q_ε^{-1},*

(ii) *the r.v.'s $v_{j,\varepsilon}$ and $\Delta X_{j,\varepsilon} = X(t_{j,\varepsilon}(-) - X(t_{j-1,\varepsilon})$, $j \geq 1$, are independent with m.g.f.'s*

$$\mathbf{E} \exp(-u\Delta(X_{j,\varepsilon}) = \left[1 + q_\varepsilon^{-1} \int_0^\varepsilon (1 - \mathrm{e}^{-uv})\, \mathrm{d}Q(v)\right]^{-1}, \quad u \geq 0,$$

$$\mathbf{E} \exp(-uv_{j,\varepsilon}) = q_\varepsilon^{-1} \int_\varepsilon^\infty \mathrm{e}^{-uv}\, \mathrm{d}Q(v), \quad u \geq 0.$$

Proof. (i) This is a well-known result concerning the waiting times of a Poisson process, see [10, p. 548].

(ii) From [10, pp. 550–552], X may be represented as the sum of two independent processes in S_1, $X = Y_1 + Y_2$, where Y_1 has Lévy measure $Q_1(v) = Q(v)I(v < \varepsilon)$, Y_2 has Lévy measure $Q_2(v) = Q(v)I(v \geq \varepsilon)$ and both have time-scale function α. Furthermore $t_{j,\varepsilon}$, $v_{j,\varepsilon}$, $j \geq 1$, are the jump times and jump sizes of Y_2, and $\Delta X_{j,\varepsilon} = Y_1(t_{j,\varepsilon}) - Y_1(t_{j-1,\varepsilon})$. Thus, $v_{j,\varepsilon}, j \geq 1$, are i.i.d. r.v.'s with d.f. $F(v) = q_\varepsilon^{-1}(q_\varepsilon - q_v)$, $v \geq \varepsilon$, and $\Delta\alpha_{j,\varepsilon}$, $v_{j,\varepsilon}$, Y_1 are mutually independent, whence

$$\log \mathbf{E}(\exp(-u\Delta X_{j,\varepsilon}) \,|\, t_{j,\varepsilon}, t_{j-1,\varepsilon}) = -\Delta\alpha_{j,\varepsilon} \int_0^\varepsilon (1 - \mathrm{e}^{-uv})\, \mathrm{d}Q(v)$$

and

$$\mathbf{E} \exp(-u\Delta X_{j,\varepsilon}) = \left[1 + q_\varepsilon^{-1} \int_0^\varepsilon (1 - \mathrm{e}^{-uv})\, \mathrm{d}Q(v)\right]^{-1}.$$

If X has Lévy measure Q, denote by X_0 the process in S_1 with Lévy measure Q and time-scale function $\alpha(t) \equiv t$. Also, define the function $H(x)$ by

$$H(x) = q_\varepsilon^{-1} \sum_{j=1}^\infty \mathbf{P}(X(t_{j,\varepsilon}-) \leq x), \quad \varepsilon > 0, \ x \geq 0.$$

Lemma 2. *If $X \in S_1$ and if F_s denotes the d.f. of $X_0(s)$, then*

$$H(x) = \int_0^\infty F_s(x)\, \mathrm{d}s. \tag{8}$$

Proof. The result follows by using Lemma 1 to show that both sides of (8) have the same Laplace transform.

Note that Lemma 2 implies that $H(x)$ does not depend on ε. Also, since X_0 has stationary increments, it can be seen that H is a delayed renewal function, $H = \sum_{j=0}^\infty F * F_1^{*j}$, where $F(x) = \int_0^1 F_s(x)\, \mathrm{d}s$.

Algorithm 1 provides a computational formula for the expectation of various random variables associated with the joint failure time vector in a wear model. In particular, the probability of obtaining a specific tie configuration (Theorem 1) can be obtained. Further applications of Algorithm 1 follow. Let $A_n = \lim_{\varepsilon\downarrow 0} A_n(\varepsilon)$, where

$$A_n(\varepsilon) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_m=1}^{\infty} A_n(\boldsymbol{j}_m; \varepsilon),$$

$\boldsymbol{j}_m = (j_1, \ldots, j_m)$ and m is a fixed positive integer. The indices $\boldsymbol{j}_m$ refer to jumps of X at which specified events occur. Let $n(i) = \sum_{r=1}^{i} j_r$, $1 \leqslant i \leqslant m$, $n(0) = 0$, and let $\Delta_i = X(t_{n(i),\varepsilon}) - X(t_{n(i-1),\varepsilon})$.

Algorithm 1. If $A_n(\boldsymbol{j}_m; \varepsilon)$ is an r.v. whose conditional distribution given $v_{n(i),\varepsilon}$, Δ_i, $1 \leqslant i \leqslant m$, does not depend on $\boldsymbol{j}_m$ and ε, and if $\mathbf{E}A_n = \lim_{\varepsilon\downarrow 0} \mathbf{E}A_n(\varepsilon)$, then

$$\mathbf{E}A_n = \int_0^{\infty} \cdots \int_0^{\infty} g_n(\boldsymbol{u}_m, \boldsymbol{v}_m) \prod_{i=1}^{m} \mathrm{d}H(u_i)\, \mathrm{d}Q(v_i) \tag{9}$$

where $g_n(\boldsymbol{u}_m, \boldsymbol{v}_m) = \mathbf{E}(A_n(\boldsymbol{j}_m; \varepsilon) \mid \Delta_i = u_i,\ v_{n(i),\varepsilon} = v_i,\ 1 \leqslant i \leqslant m)$.

Proof. The proof is exhibited for $m = 1$. From Lemma 2,

$$\begin{aligned}\mathbf{E}A_n &= \lim_{\varepsilon\downarrow 0} \sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \int_0^{\infty} g_n(u, v)\, \mathrm{d}F_{j,\varepsilon}(u) q_\varepsilon^{-1}\, \mathrm{d}Q(v) \\ &= \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\infty} \int_0^{\infty} g_n(u, v)\, \mathrm{d}H(u)\, \mathrm{d}Q(v) \\ &= \int_0^{\infty} \int_0^{\infty} g_n(u, v)\, \mathrm{d}H(u)\, \mathrm{d}Q(v).\end{aligned}$$

The proof for $m \geqslant 2$ is essentially the same and uses the additional fact (from Lemma 1) that Δ_i, $1 \leqslant i \leqslant m$, are independent with d.f. $F_{j_i,\varepsilon}$.

Define R_+^n to be the set of all n-tuples $\boldsymbol{t} = (t_1, \ldots, t_n)$ such that $t_j > 0$, and define $C_{n,k}$, $1 \leqslant k \leqslant n$, to be the set of all k-tuples $\boldsymbol{c} = (c_1, \ldots, c_k)$ of positive integers such that $\sum_{j=1}^{k} c_j = n$. For $\boldsymbol{c} \in C_{n,k}$, define $C^{-1}(\boldsymbol{c}) = \{\boldsymbol{t} \in R_+^n : C(\boldsymbol{t}) = \boldsymbol{c}\}$. Theorem 1 gives the probability of obtaining a specific tie configuration.

Theorem 1. *In a wear model, for each* $\boldsymbol{c} \in C_{n,k}$, $1 \leqslant k \leqslant n$,

$$\mathbf{P}(C^{-1}(\boldsymbol{c})) = \binom{n}{\boldsymbol{c}} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^{k} (\Delta \bar{G}_j)^{c_j}\, \mathrm{d}H(u_j)\, \mathrm{d}Q(v_j)$$

where

$$\Delta \bar{G}_j = \bar{G}\Big(\sum_{i=1}^{j} (u_i + v_{i-1})\Big) - \bar{G}\Big(\sum_{i=1}^{j} (u_i + v_i)\Big), \quad v_0 = 0$$

and

$$\binom{n}{c} = n!(c_1! \cdots c_k!)^{-1}.$$

Proof. The proof follows from Algorithm 1 by setting

$$A_n(j_k;\varepsilon) = \prod_{j=1}^{k} I(\text{exactly } c_i \text{ devices fail at } t_{n(i),\varepsilon}).$$

Corollary 1. *In a Gaver model, for each* $c = C_{n,k}$, $1 \leqslant k \leqslant n$,

$$\mathbf{P}(C^{-1}(c)) = \binom{n}{c} \prod_{j=1}^{k} [M(s_j)]^{-1} \int_0^\infty (1-e^{-v})^{c_i} \exp(-s_{j+1}v)\, dQ(v).$$

Corollary 1 is Theorem 1.16 of [12]. Note that the results of Theorem 1 and Corollary 1 can be used to test hypotheses concerning parameters of the Lévy measure. Corollary 2 can be used as the basis of an estimation procedure for a parameter of the Lévy measure.

Corollary 2. *In a Gaver model, if j is a fixed positive integer, and*

(i) *if* $q_0 < \infty$, *then* $a_j = s_{j+1}/s_j$ *has a limiting distribution with m.g.f. given by*

$$\lim_{n\to\infty} \mathbf{E}\exp(-ua_j) = q_0^{-1} \int_0^\infty \exp(-u\, e^{-v})\, dQ(v), \quad u \geqslant 0,$$

(ii) *if* $q_0 = \infty$, *then* $a_j \to 1$ (p), *where* (p) *denotes convergence in probability.*

Proof. Through some elementary computations with (9), it can be shown that

$$\mathbf{P}(s_{j-1} = m \mid s_j) = \binom{s_j}{m} [M(s_j)]^{-1} \int_0^\infty e^{-mv}(1-e^{-v})^{s_j-m}\, dQ(v),$$

$0 \leqslant m \leqslant s_j$, $1 \leqslant j \leqslant k$. The desired results follow from induction and the dominated convergence theorem. Note that the results hold simultaneously for all $1 \leqslant i \leqslant j$.

Results for K_n, the number of distinct failure times, and N_m, the number of times exactly m devices fail, are given in Theorem 2. Note that $K_n = \sum_{m=1}^{n} N_m$.

Theorem 2. *In a wear model,*

$$\mathbf{E}N_m = \binom{n}{m} \int_0^\infty \int_0^\infty [\bar{G}(u) - \bar{G}(u+v)]^m [1 - \bar{G}(u) + \bar{G}(u+v)]^{n-m}\, dH(u)\, dQ(v),$$

$$\mathbf{E}K_n = \int_0^\infty \int_0^\infty (1 - [1 - \bar{G}(u) + \bar{G}(u+v)]^n)\, dH(u)\, dQ(v).$$

Large sample results for some of these r.v.'s can be obtained through Algorithms 2 and 3. The following notation is used:

$$P(n)=\int_0^\infty \int_0^\infty g_n(u, v)\,\mathrm{d}H(u)\,\mathrm{d}Q(v), \tag{10}$$

$$R(n)=\int_0^\infty \int_0^\infty g_n(u, v)\,\mathrm{d}u\,\mathrm{d}Q(v), \tag{11}$$

$$\beta=\int_0^\infty v\,\mathrm{d}Q(v). \tag{12}$$

Note that since H is a renewal function, then $H(u)/u \to \beta^{-1}$ as $u \to \infty$, see [6, p. 347].

Algorithm 2. If $X \in S_1$ and if $g_n(u, v)$ is a nonnegative function which is nonincreasing in u and satisfies

(i) $R(n)<\infty$,
(ii) $(\partial/\partial u)g_n(u, v)=g_n'(u, v)$ exists, $u, v>0$,
(iii) $\lim_{u\to\infty} u g_n(u, v)=g_n(v)$ exists and $\int_0^\infty g_n(v)\,\mathrm{d}Q(v)<\infty$,
(iv) $\int_0^\infty g_n(0, v)\,\mathrm{d}Q(v)=\mathrm{o}(R(n))$ as $n\to\infty$,

then $P(n)=\beta^{-1}R(n)+\mathrm{o}(R(n))$ as $n\to\infty$.

Proof. Integrate (10) and (11) by parts to obtain

$$\begin{aligned} P(n)=\beta^{-1}&\int_0^\infty g_n(v)\,\mathrm{d}Q(v)-H(0)\int_0^\infty g_n(0, v)\,\mathrm{d}Q(v)\\ &-\int_0^\infty \int_0^\infty g_n'(u, v)H(u)\,\mathrm{d}u\,\mathrm{d}Q(v), \end{aligned} \tag{13}$$

$$R(n)=\int_0^\infty g_n(v)\,\mathrm{d}Q(v)-\int_0^\infty \int_0^\infty g_n'(u, v)u\,\mathrm{d}u\,\mathrm{d}Q(v). \tag{14}$$

Let $\varepsilon>0$ and let z be sufficiently large so that $|H(u)/u-\beta^{-1}|<\varepsilon$ for $u>z$. Then

$$\begin{aligned} |P(n)-\beta^{-1}R(n)| &\le H(0)\int_0^\infty g_n(0, v)\,\mathrm{d}Q(v)\\ &\quad+\int_0^\infty \int_0^\infty -g_n'(u, v)|H(u)-u\beta^{-1}|\,\mathrm{d}u\,\mathrm{d}Q(v)\\ &\le [H(0)+H(z)+z\beta^{-1}]\int_0^\infty g_n(0, v)\,\mathrm{d}Q(v)\\ &\quad+\varepsilon\int_0^\infty \int_z^\infty -g_n'(u, v)u\,\mathrm{d}u\,\mathrm{d}Q(v)\\ &\le [H(0)+H(z)+z\beta^{-1}+\varepsilon z]\int_0^\infty g_n(0, v)\,\mathrm{d}Q(v)+\varepsilon R(n), \end{aligned}$$

which implies the desired result.

Note that under the conditions of Algorithm 2 it is not necessary to compute H to obtain the asymptotic behavior of $P(n)$. The asymptotic behavior of $R(n)$ can be determined in many cases with Lemma 3. The proof is elementary and so is omitted.

Lemma 3. *If $\{x_j\}$ is a sequence of positive real numbers and if f is a real-valued function which satisfies*

(i) *f has a positive derivative f' on $[a, \infty)$ for some $0<a<\infty$ and f is ultimately monotone,*

(ii) *$\lim_{n\to\infty} x_n/f'(n) = x$ for some $0 \leq x < \infty$,*

(iii) *$f(n)\to\infty$ and $f'(n) = 0(1)$ as $n\to\infty$, then*

$$\lim_{n\to\infty} [f(n)]^{-1} \sum_{j=1}^{n} x_j = x.$$

Theorem 3. *In a Gaver model, if $q_0 < \infty$, then*

$$\mathbf{E}K_n = \beta^{-1} q_0 \log n + \mathrm{o}(\log n) \quad \text{as } n\to\infty.$$

Proof. Note that $R(n) = \sum_{j=1}^{n} j^{-1} M(j)$. Since $\lim_{n\to\infty} M(n) = q_0$, apply Lemma 3 with $f(n) = \log n$ and $x = q_0$ to obtain the desired result.

Theorem 4. *In a Gaver model, if X has Lévy measure of the form*

$$Q(v) = -\int_v^\infty x^{-1} \mathrm{e}^{-x}\, \mathrm{d}x + Q_1(v),$$

and if $Q_1(0) > -\infty$, then

$$\mathbf{E}K_n = (2\beta)^{-1} \log^2 n + \mathrm{o}(\log^2 n) \quad \text{as } n\to\infty.$$

Proof. Note that $M(n) = \log n + \mathrm{o}(1)$. Apply Lemma 3 with $f(n) = 2^{-1}\log^2 n$ and Algorithm 2 to obtain the desired result.

A process which satisfies the conditions of Theorem 4 is called a *gamma-type process*. In particular, a gamma process satisfies these conditions.

The following notation is used with Algorithm 3:

$$Z(n) = 2\int_0^\infty \cdots \int_0^\infty g_n(u, v) g_n(u+x, y)\, \mathrm{d}x\, \mathrm{d}Q(y)\, \mathrm{d}H(u)\, \mathrm{d}Q(v),$$

$$W(n) = 2\int_0^\infty \cdots \int_0^\infty g_n(u, v) g_n(u+x, y)\, \mathrm{d}H(x)\, \mathrm{d}Q(y)\, \mathrm{d}H(u)\, \mathrm{d}Q(v).$$

Note that

$$R^2(n) = 2\int_0^\infty \cdots \int_0^\infty g_n(u, v) g_n(u+x, y)\, \mathrm{d}x\, \mathrm{d}Q(y)\, \mathrm{d}u\, \mathrm{d}Q(v).$$

Because of its similarity to Algorithm 2, the proof of Algorithm 3 is omitted.

Algorithm 3. If $g_n(u, v)$ satisfies the conditions of Algorithm 2, then

$$Z(n)=\beta^{-1}R^2(n)+\mathrm{o}(R^2(n)) \quad \text{and} \quad W(n)=\beta^{-2}R^2(n)+\mathrm{o}(R^2(n)) \quad \text{as } n\to\infty.$$

Theorem 5. *In a Gaver model,*

(i) *if* $q_0<\infty$, *then* $K_n/\log n\to\beta^{-1}q_0$ (p),

(ii) *if* X *is a gamma-type process, then* $K_n/\log^2 n\to(2\beta)^{-1}$ (p).

Proof. Apply Algorithm 1 to K_n^2 to obtain

$$\mathbf{E}K_n^2=\mathbf{E}K_n+2\int_0^\infty\cdots\int_0^\infty f_n(u, v, x, y)\,\mathrm{d}H(x)\,\mathrm{d}Q(y)\,\mathrm{d}H(u)\,\mathrm{d}Q(v),$$

where

$$f_n(u, v, x, y)=1-[1-\mathrm{e}^{-u}(1-\mathrm{e}^{-v})]-[1-\mathrm{e}^{-u-v-x}(1-\mathrm{e}^{-y})]^n$$
$$+[1-\mathrm{e}^{-u}(1-\mathrm{e}^{-v})-\mathrm{e}^{-u-v-x}(1-\mathrm{e}^{-y})]^n.$$

Since $f_n(u, v, x, y)\leq g_n(u, v)g_n(u+x, y)$, then from Algorithm 3,

$$\mathbf{E}K_n^2\leq\beta^{-2}R^2(n)+\mathrm{o}(R^2(n)).$$

Thus, var $K_n=\mathrm{o}(R^2(n))$ which implies the required results.

Example 1. Let X be a compound Poisson process with a Levy measure which is absolutely continuous with density $q(v)=\beta^{-1}\,\mathrm{e}^{-v/\beta}$, $\beta>0$. Then $q_0=1$ and the jump sizes of X are i.i.d. with an exponential distribution. Note that $M(u)=u\beta(1+u\beta)^{-1}$. In a Gaver model, $K_n/\log n\to\beta^{-1}$ from Theorem 5. Also, it can be seen that $H(x)=1+\beta^{-1}x$. Thus in a Gaver model, the probability of obtaining a specific tie configuration is, from Corollary 1,

$$\mathbf{P}(C^{-1}(c))=n!\,\beta^{-2k}\prod_{j=1}^{k}s_j^{-1}(1+s_j\beta)\Gamma(s_{j+1}+\beta^{-1})/\Gamma(s_j+\beta^{-1}+1).$$

The results given here illustrate the use of the algorithms to obtain probabilistic properties of the joint failure time vector. These algorithms are used in [2] to obtain estimates of a parameter of the Lévy measure with simultaneous life testing.

References

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